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Quasi-exactly solvable potentials with three known eigenstates

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Abstract. We propose a new supersymmetric (SUSY) method for the generation of the quasi-exactly solvable (QES) potentials with three known eigenstates. New QES potentials and corresponding energy levels and wavefunctions of the ground state and two lowest excited states are obtained. The possibility of constructing families of exactly solvable non-singular potentials which are SUSY partners of the well known ones is shown.

1. Introduction

About 20 years ago an interesting class of the so-called quasi-exactly solvable (QES) potentials for which a finite number of eigenstates is analytically known was introduced [1–4]. Nowadays, the QES problems attract much attention [5–19]. Several methods for generation of QES potentials have been worked out and as a result many QES potentials have been established. For example, three different methods based respectively on a polynomial ansatz for wavefunctions, point canonical transformation and supersymmetric (SUSY) quantum mechanics are described in [12].

The SUSY method is a very useful tool for the study of exactly solvable potentials. Note the papers [20, 21] where a SUSY procedure for constructing Hamiltonians either with identical spectra or with identical spectra, apart from a missing ground state, was given. This procedure may be repeated again and again to generate hierarchies of Hamiltonians whose spectra are related to each other. One can find some recent papers on this subject in [22–25]. For a review of SUSY quantum mechanics see [26, 27].

The SUSY method for constructing QES potentials was used for the first time in [10–12]. The idea of this method is as follows. Starting from some initial QES potential with $n + 1$ known eigenstates and using the properties of the unbroken supersymmetric one obtains the SUSY partner potential which is a new QES one with n known eigenstates.

In our previous paper [28] we proposed a new SUSY method for generating QES potentials with two eigenstates which are explicitly known. This method, in contrast to those of [10–12], does not require knowledge of the initial QES potential in order to generate a new QES one. In the present paper we develop this SUSY method to construct QES potentials with three eigenstates which are explicitly known.

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2. The Witten model of SUSY quantum mechanics

Let us first take a look at the Witten model of SUSY quantum mechanics. The algebra of supersymmetry in this case satisfies the following permutation relations

$$\{Q^+, Q^-\} = H \qquad [Q^\pm, H] = 0 \qquad (Q^\pm)^2 = 0 \tag{1}$$

where the supercharges read

$$Q^+ = B^-\sigma^+ \qquad Q^- = B^+\sigma^- \tag{2}$$

$\sigma^\pm$ are the Pauli matrices,

$$B^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right) \tag{3}$$

and $W(x)$ is the superpotential. The Hamiltonian consists of a pair of standard Schrödinger operators $H_\pm$

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{4}$$

where

$$H_\pm = B^\mp B^\pm = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x) \tag{5}$$

and $V_\pm(x)$ are the so-called SUSY partner potentials

$$V_\pm(x) = \frac{1}{2}(W^2(x) \pm W'(x)) \qquad W'(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{6}$$

Consider the equation for the energy spectrum

$$H_\pm\psi_n^\pm(x) = E_n^\pm\psi_n^\pm(x) \qquad n = 0, 1, 2, \ldots. \tag{7}$$

The Hamiltonians H_+ and H_- have the same energy spectrum except for the zero energy ground state which exists in the case of the unbroken supersymmetry. This leads to twofold degeneracy of the energy spectrum of H, except for the unique zero-energy ground state. Only one of the Hamiltonians $H_\pm$ has the zero-energy eigenvalue. We shall use the convention that the zero-energy eigenstate belongs to H_-. Due to the factorization of the Hamiltonians $H_\pm$ (see (5)) the ground state for H_- satisfies the equation

$$B^-\psi_0^-(x) = 0 \tag{8}$$

with the solution

$$\psi_0^-(x) = C\exp\left(-\int W(x)\,\mathrm{d}x\right) \tag{9}$$

where C is the normalization constant. From the normalization condition it follows that

$$\mathrm{sign}(W(x)) = \pm 1 \tag{10}$$

when $x \to \pm\infty$.

The eigenvalues and eigenfunctions of the Hamiltonians H_+ and H_- are related by the SUSY transformations

$$E_{n+1}^- = E_n^+ \qquad E_0^- = 0 \tag{11}$$

$$\psi_{n+1}^-(x) = \frac{1}{\sqrt{E_n^+}}B^+\psi_n^+(x) \tag{12}$$

$$\psi_n^+(x) = \frac{1}{\sqrt{E_{n+1}^-}}B^-\psi_{n+1}^-(x). \tag{13}$$

The two properties of the unbroken SUSY quantum mechanics, namely, a twofold degeneracy of the spectrum and the existence of the zero-energy ground state, are used for the exact calculation of the energy spectrum and wavefunctions (see reviews [26, 27]).

3. SUSY constructing QES potentials

We shall study the Hamiltonian H_-, the ground state of which is given by (9). Let us consider the SUSY partner of H_-, i.e. the Hamiltonian H_+. If we calculate the ground state of H_+ we immediately find the first excited state of H_- using the degeneracy of the spectrum of the SUSY Hamiltonian and transformations (11)–(13). In order to calculate the ground state of H_+ let us rewrite it in the following form

$$H_+ = H_-^{(1)} + \epsilon \qquad \epsilon > 0 \tag{14}$$

where

$$H_-^{(1)} = B_1^+ B_1^- \qquad B_1^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{\mathrm{d}}{\mathrm{d}x} + W_1(x)\right) \tag{15}$$

and ϵ is the energy of the ground state of H_+ since $H_-^{(1)}$ has zero-energy ground state.

As we see from (14) and (15), the ground-state wavefunction of H_+ is also the ground-state wavefunction of $H_-^{(1)}$ and it satisfies the equation

$$B_1^- \psi_0^+(x) = 0. \tag{16}$$

The solution of this equation is

$$\psi_0^+(x) = C \exp\left(-\int W_1(x)\,\mathrm{d}x\right). \tag{17}$$

Then using the SUSY transformation (12) we can easily calculate the wavefunction of the first excited state of H_-. Repeating the described procedure for $H_-^{(1)}$ we obtain the second excited state for H_-. Continuing this procedure N times we obtain N excited states. This procedure is well known in SUSY quantum mechanics [20, 21] (see also reviews [26, 27]). The wavefunctions and corresponding energy levels read

$$\psi_n^-(x) = C_n^- B_0^+ \dots B_{n-2}^+ B_{n-1}^+ \exp\left(-\int W_n(x)\,\mathrm{d}x\right) \tag{18}$$

$$E_n^- = \sum_{i=0}^{n-1} \epsilon_i \tag{19}$$

where $n = 1, 2, \dots, N$. In our notation $\epsilon_0 = \epsilon$, $B_0^{\pm} = B^{\pm}$, $W_0(x) = W(x)$. Operators $B_n^{\pm}$ are given by (3) with the superpotentials $W_n(x)$. The equation (14) rewritten for N steps

$$H_+^{(n)} = H_-^{(n+1)} + \epsilon_n \qquad n = 0, 1, \dots, N-1 \tag{20}$$

leads to the set of equations for superpotentials

$$W_n^2(x) + W_n'(x) = W_{n+1}^2(x) - W_{n+1}'(x) + 2\epsilon_n \qquad n = 0, 1, \dots, N-1. \tag{21}$$

Previously this set of equations for $W_n(x)$ was solved in the special cases of the so-called shape-invariant potentials [29] and self-similar potentials for arbitrary N (see review [30]). For $N = 1$ one can easily obtain a general solution of (21) without restricting ourselves to shape-invariant or self-similar potentials. This solution was obtained in [31] in the context of parasupersymmetric quantum mechanics.

In our recent paper [28] we constructed a non-singular solution of (21) for $N = 1$ in order to obtain non-singular QES potentials with two known eigenstates.

In the present paper we use the method proposed in [28] to solve the set of equations (21) for $N = 2$. This gives us the possibility of obtaining the general expression for wavefunctions and energy levels of QES potentials with three eigenstates which are explicitly known.

Let us introduce the new functions

$$W_+^{(n)}(x) = W_{n+1}(x) + W_n(x)$$
$$W_-^{(n)}(x) = W_{n+1}(x) - W_n(x) \qquad n = 0, 1, \ldots, N-1. \tag{22}$$

Then equations (21) read

$$W_+'^{(n)}(x) = W_-^{(n)}(x) W_+^{(n)}(x) + 2\epsilon_n. \tag{23}$$

One can easily solve these equations with respect to the functions $W_-^{(n)}(x)$ and obtain the following expressions for superpotentials

$$W_n(x) = \frac{1}{2}\left(W_+^{(n)}(x) - \frac{W_+'^{(n)}(x) - 2\epsilon_n}{W_+^{(n)}(x)}\right)$$
$$W_{n+1}(x) = \frac{1}{2}\left(W_+^{(n)}(x) + \frac{W_+'^{(n)}(x) - 2\epsilon_n}{W_+^{(n)}(x)}\right) \tag{24}$$

which lead obviously to the following set of equations for the functions $W_+^{(n)}(x)$

$$W_+^{(n)}(x) + \frac{W_+'^{(n)}(x) - 2\epsilon_n}{W_+^{(n)}(x)} = W_+^{(n+1)}(x) + \frac{W_+'^{(n+1)}(x) - 2\epsilon_{n+1}}{W_+^{(n+1)}(x)} \qquad n = 0, 1, \ldots, N-2. \tag{25}$$

Thus, the set of N equations (21) is reduced to the set of $N-1$ equations (25). In the simplest case $N = 1$ equation (25) is absent and relations (24) express just a general solution of equation (21). To obtain a general solution of the set (21) in the case $N = 2$ we have to solve one equation,

$$W_+(x) + \frac{W_+'(x) - 2\epsilon}{W_+(x)} = \tilde{W}_+(x) - \frac{\tilde{W}_+'(x) - 2\epsilon_1}{\tilde{W}_+(x)} \tag{26}$$

where we have introduced for simplicity the notation

$$W_+(x) \equiv W_+^{(0)}(x) \qquad \tilde{W}_+(x) \equiv W_+^{(1)}(x). \tag{27}$$

It is easy to rewrite this equation as follows

$$W_+(x)\tilde{W}_+(x)(\tilde{W}_+(x) - W_+(x)) - (W_+(x)\tilde{W}_+(x))' + 2(\epsilon_1 W_+(x) + \epsilon \tilde{W}_+(x)) = 0 \tag{28}$$

or

$$U(x)\left(\frac{U(x)}{W_+(x)} - W_+(x)\right) - U'(x) + 2\left(\epsilon_1 W_+(x) + \epsilon \frac{U(x)}{W_+(x)}\right) = 0 \tag{29}$$

where we have introduced the function

$$U(x) = W_+(x)\tilde{W}_+(x). \tag{30}$$

We again arrive at the Riccati equation with respect to $U(x)$. On the other hand, this is an algebraic equation with respect to $W_+(x)$. Thus, we can start from an arbitrary function $U(x)$ to construct the functions $W_+(x)$ and $\tilde{W}_+(x)$ which take the form

$$W_+(x) = \frac{2U(x)(U(x) + 2\epsilon)}{U'(x)\,(1 + \mathcal{R}(x))} \qquad \tilde{W}_+(x) = \frac{U'(x)\,(1 + \mathcal{R}(x))}{2(U(x) + 2\epsilon)} \tag{31}$$

where

$$\mathcal{R}(x) = \pm R(x) \tag{32}$$
$$R(x) = \sqrt{1 + 4\frac{U(x)(U(x) + 2\epsilon)(U(x) - 2\epsilon_1)}{U'(x)^2}}. \tag{33}$$

The square root $R(x)$ is a positively defined value, while the function $\mathcal{R}(x)$ can be chosen in the form of $R(x)$ or $-R(x)$ within different intervals separated by zeros of the function $R(x)$. Note that just the possibility of being able to choose different signs allows us, as will be shown in section 5, to construct in a simple way new exactly solvable potentials using the known ones.

Now we can obtain three consequent superpotentials $W(x)$, $W_1(x)$ and $W_2(x)$ using the relations (24). Then using (18) and (19) we obtain the energy levels and the wavefunctions of the first and the second excited states for H_-

$$E_1^- = \epsilon \qquad E_2^- = \epsilon + \epsilon_1 \tag{34}$$

$$\psi_1^-(x) = C_1 W_+(x) \exp\left(-\int W_1(x)\,\mathrm{d}x\right)$$

$$\psi_2^-(x) = C_2((W(x) + W_2(x))\tilde{W}_+(x) - \tilde{W}_+') \exp\left(-\int W_2(x)\,\mathrm{d}x\right). \tag{35}$$

The superpotentials $W_1(x)$ and $W_2(x)$ must satisfy the same condition (10) as $W(x)$. This leads to the same limitations for the functions $W_+(x)$ and $\tilde{W}_+(x)$. Both must be positive at infinity, negative at minus infinity and therefore each of them must possess at least one zero. Let us consider first the continuous superpotentials. As is seen from (24), to avoid singularity of the superpotentials the functions $W_+(x)$ and $\tilde{W}_+(x)$ each ought to have only one zero [28]

$$W_+(x_0) = 0 \qquad \tilde{W}_+(\tilde{x}_0) = 0$$

at which they must satisfy the conditions

$$W_+'(x_0) = 2\epsilon \qquad \tilde{W}_+'(\tilde{x}_0) = 2\epsilon_1. \tag{36}$$

Thus we have a number of limitations in the choice of the function $U(x)$ as a product of $W_+(x)$ and $\tilde{W}_+(x)$. There are two different possibilities for the choice of the function $U(x)$. Either $x_0 = \tilde{x}_0$ which means that $U(x)$ has only one second-order zero point and is positive along the rest of the number line

$$\begin{cases} U(x_0) = 0 \qquad U'(x_0) = 0 \qquad U''(x_0) > 0 \\ U(x) > 0 \qquad x \neq x_0 \end{cases} \tag{37}$$

or $x_0 \neq \tilde{x}_0$ and therefore $U(x)$ has two zero points and changes its sign as follows

$$\begin{cases} U(x) < 0 & x \in (\min[x_0, \tilde{x}_0], \max[x_0, \tilde{x}_0]) \\ U(x_0) = U(\tilde{x}_0) = 0 & U'(\min[x_0, \tilde{x}_0]) < 0, U'(\max[x_0, \tilde{x}_0]) > 0 \\ U(x) > 0 & x \notin [\min[x_0, \tilde{x}_0], \max[x_0, \tilde{x}_0]]. \end{cases} \tag{38}$$

The sign of the function $\mathcal{R}(x)$ in expressions (31) for the functions $W_+(x)$ and $\tilde{W}_+(x)$ should be chosen in a such way to ensure smoothness of these functions and the existence of one zero for each of them. A full analysis of the conditions which function $U(x)$ must satisfy to provide continuous superpotentials is rather boring and includes consideration of the behaviour of the functions $W_+(x)$ and $\tilde{W}_+(x)$ near possible zeros of the expressions

$$U(x) + 2\epsilon \qquad U(x) - 2\epsilon_1 \qquad U'(x) \qquad R(x) \tag{39}$$

which is crucial for the continuity of the final superpotentials $W(x)$, $W_1(x)$ and $W_2(x)$.

We shall consider more closely the simplest case of the function $U(x)$ which has one zero point and satisfies the conditions (37). The other condition which the function $U(x)$ must satisfy is a consequence of the conditions (36) that connect the derivatives of the functions $W_+(x)$ and $\tilde{W}_+(x)$ with the energies ϵ and ϵ_1. One can easily find that

$$\begin{aligned} U''(x_0) &= W_+''(x_0)\tilde{W}_+(x_0) + 2W_+'(x_0)\tilde{W}_+'(x_0) + W_+(x_0)\tilde{W}_+''(x_0) \\ &= 2W_+'(x_0)\tilde{W}_+'(x_0) = 8\epsilon\epsilon_1. \end{aligned} \tag{40}$$

The other obvious condition imposed on the function $U(x)$ is positivity of the expression under the square root of the function $R(x)$, (33)

$$\frac{U'(x)^2 + 4U(x)(U(x)+2\epsilon)(U(x)-2\epsilon_1)}{U'(x)^2} \geqslant 0. \tag{41}$$

One can easily check that $R(x_0) = 0$. Let us consider the case when the point x_0 is a unique zero of the function $R(x)$. Then the only way to construct non-singular potentials is to chose the function $\mathcal{R}(x) = R(x)$ over all the line in (32). Moreover, we shall require the function $R(x)$ to be smooth in the vicinity of the point x_0 to avoid cusps of the functions $W_+(x)$ and $\tilde{W}_+(x)$ at this point. Note that such cusps would result in δ-like singularities of the final potential $V_-(x)$.

Thus we obtain one of the possible sets of conditions for the function $U(x)$ allowing us to construct non-singular QES potentials with three known eigenstates

$$\begin{aligned} & U(x) > 0 \qquad \forall x \neq x_0 \\ & U(x_0) = 0 \qquad U'(x_0) = 0 \qquad U''(x_0) = 8\epsilon\epsilon_1 \qquad U'''(x_0) = 0 \\ & U^{(4)}(x_0) = 64\epsilon\epsilon_1(\epsilon_1 - \epsilon) \qquad U^{(5)}(x_0) = 0 \qquad \frac{U^{(6)}(x_0)}{8\epsilon\epsilon_1} \geqslant 32(2\epsilon_1^2 - 13\epsilon\epsilon_1 + 2\epsilon^2) \\ & R(x) > 0 \qquad \forall x \neq x_0. \end{aligned} \tag{42}$$

Note that most of the exactly solvable potentials which are continuous over all the line satisfy these conditions.

4. Examples

One can easily check that the simplest functions $U(x)$ yield the well known potentials. For example, starting from $U(x) = 4\epsilon\epsilon_1 x^2$ at $\epsilon_1 = \epsilon$ we obtain the harmonic oscillator potential. Another simple function $U(x) = 4\epsilon\epsilon_1 \tanh^2 x$ at $\epsilon_1 = \epsilon - 1$ leads us to the well known exactly solvable Rosen–Morse potential. One more simple example $U(x) = 4\epsilon\epsilon_1 \sinh^2 x$ at $\epsilon_1 = \epsilon + \frac{1}{2}$ reproduces the special case of the well known quasi exactly solvable Razavy potential [3].

Let us consider more complicated examples leading to new QES potentials. We shall start from the function

$$U(x) = 4\epsilon\epsilon_1 x^2 \frac{1 + a^2x^2}{1 + b^2x^2} \tag{43}$$

where a and b are real parameters.

Due to the conditions for $U^{(4)}(x_0)$ and $U^{(6)}(x_0)$ from (42) we get

$$a^2 = b^2 + \tfrac{2}{3}(\epsilon_1 - \epsilon) \tag{44}$$

and

$$\frac{13}{4}\frac{\epsilon}{b^2} - \frac{15}{8} - \frac{3}{8}\sqrt{\Delta} \leqslant \frac{\epsilon_1}{b^2} \leqslant \frac{13}{4}\frac{\epsilon}{b^2} - \frac{15}{8} + \frac{3}{8}\sqrt{\Delta} \tag{45}$$

respectively; here

$$\Delta = 25 - 60\frac{\epsilon}{b^2} + 68\left(\frac{\epsilon}{b^2}\right)^2. \tag{46}$$

Because a and b are real numbers it follows from (44) that

$$\frac{\epsilon}{b^2} - \frac{3}{2} \leqslant \frac{\epsilon_1}{b^2}. \tag{47}$$

It is easy to check that

$$\frac{13}{4}\frac{\epsilon}{b^2} - \frac{15}{8} - \frac{3}{8}\sqrt{\Delta} \leqslant \frac{\epsilon}{b^2} - \frac{3}{2} \tag{48}$$

and, therefore, (45) together with (44) lead to the following inequality:

$$\frac{\epsilon}{b^2} - \frac{3}{2} \leqslant \frac{\epsilon_1}{b^2} \leqslant \frac{13}{4}\frac{\epsilon}{b^2} - \frac{15}{8} + \frac{3}{8}\sqrt{\Delta}. \tag{49}$$

Thus, in order to obtain non-singular potentials the parameters of function (43) must satisfy conditions (44) and (49). Moreover, we obviously must require the parameters ϵ and ϵ_1 to be positive.

We shall omit the general expression for the potential $V_-(x)$ as it is huge and rather useless. It is easy to show that there are only two sets of positive ϵ and ϵ_1 which allow us to resolve the root in the function $R(x)$ (cases 1 and 2) and therefore to simplify significantly the final expressions. The other simplified expression (case 3) we shall obtain by putting $a = 0$ that corresponds to the lowest value of the parameter ϵ_1 in inequality (49).

4.1. *Case 1*

In the case

$$\epsilon = \tfrac{3}{2}b^2 \qquad \epsilon_1 = (\tfrac{3}{2} + \sqrt{3})b^2 \tag{50}$$

the square root in $R(x)$ can be resolved and we obtain the potential

$$V_-(x) = \left(-\frac{9}{4} - 2\sqrt{3} + 3\left(\frac{7}{8} + \frac{\sqrt{3}}{2}\right)b^2x^2 + \frac{3-\sqrt{3}}{2-\sqrt{3}+b^2x^2} + \frac{7\sqrt{3}-12}{(2-\sqrt{3}+b^2x^2)^2}\right)b^2 \tag{51}$$

and the eigenfunctions

$$\psi_0(x) = C\,\mathrm{e}^{-(3+2\sqrt{3})b^2x^2/4}(2-\sqrt{3}+b^2x^2)^{(3-\sqrt{3})/2} \tag{52}$$

$$\psi_1(x) = C_1\,\mathrm{e}^{-(3+2\sqrt{3})b^2x^2/4}(2-\sqrt{3}+b^2x^2)^{(\sqrt{3}-1)/2}x \tag{53}$$

$$\psi_2(x) = C_2\,\mathrm{e}^{-(3+2\sqrt{3})b^2x^2/4}(2-\sqrt{3}+b^2x^2)^{(\sqrt{3}-1)/2}(2-\sqrt{3}-b^2x^2) \tag{54}$$

corresponding to the three lowest levels

$$E_0^- = 0 \qquad E_1^- = \tfrac{3}{2}b^2 \qquad E_2^- = (3+\sqrt{3})b^2. \tag{55}$$

To simplify these expressions let us make the substitution

$$b^2 = (2-\sqrt{3})c^2. \tag{56}$$

Thus, we get

$$V_-(x) = c^2\left(\frac{3}{8}c^2x^2 + \frac{3-\sqrt{3}}{1+c^2x^2} + \frac{2\sqrt{3}-3}{(1+c^2x^2)^2} - \frac{7\sqrt{3}}{4} + \frac{3}{2}\right) \tag{57}$$

$$E_0^- = 0 \qquad E_1^- = 3\left(1 - \frac{\sqrt{3}}{2}\right)c^2 \qquad E_2^- = (3-\sqrt{3})c^2 \tag{58}$$

$$\psi_0^-(x) = C\,\mathrm{e}^{-\sqrt{3}c^2x^2/4}(1+c^2x^2)^{(3-\sqrt{3})/2} \tag{59}$$

$$\psi_1^-(x) = C_1\,\mathrm{e}^{-\sqrt{3}c^2x^2/4}(1+c^2x^2)^{(\sqrt{3}-1)/2}x \tag{60}$$

$$\psi_2^-(x) = C_2\,\mathrm{e}^{-\sqrt{3}c^2x^2/4}(1+c^2x^2)^{(\sqrt{3}-1)/2}(1-c^2x^2). \tag{61}$$

It is worth noting that the obtained potential (57) is a double-well potential. As is well known, double-well potentials have been used extensively to model a wide range of natural phenomena.

4.2. Case 2

The other set of positive ϵ and ϵ_1 resolving the root in the function $R(x)$ reads

$$\epsilon = \tfrac{3}{2}b^2 \qquad \epsilon_1 = \tfrac{1}{2}b^2. \tag{62}$$

This leads to the well known supersymmetric partner of a harmonic oscillator [22, 23, 33]

$$V_-(x) = \frac{3b^2}{4} + \frac{b^4x^2}{8} - \frac{4b^2}{(1+b^2x^2)^2} + \frac{2b^2}{1+b^2x^2}. \tag{63}$$

This potential is exactly solvable although we have found it using the procedure for constructing QES potentials with three exactly known eigenstates. We will also obtain the same potential in section 5 using another procedure which allows us to construct new exactly solvable potentials (see (81)).

4.3. Case 3

Let us consider another particular case of the potential under consideration which is rather simple. We put $a = 0$ to reduce the function $U(x)$, (43), to the form

$$U(x) = 4\epsilon\epsilon_1 \frac{x^2}{1+b^2x^2}. \tag{64}$$

Due to the relation (44) we obtain the connection between the energies ϵ and ϵ_1

$$\epsilon = \epsilon_1 + \tfrac{3}{2}b^2. \tag{65}$$

Note that in accordance with relation (49) such a choice of parameters corresponds to the minimal value of ϵ_1 at given ϵ and b. It leads directly to the potential

$$\begin{aligned}\frac{V_-(x)}{b^2} = {} & \frac{(\alpha^2+3\alpha+1)^2}{8\alpha(\alpha+1)} + \frac{1}{1+b^2x^2} - \frac{2}{(1+b^2x^2)^2} - \frac{2}{\rho(x)(1+b^2x^2)^2} \\ & - \frac{\alpha^2+\alpha-1}{\rho(x)(1+b^2x^2)} - \frac{(\alpha^2+\alpha+1)^3}{8\alpha(\alpha+1)\rho^2(x)} + \frac{(\alpha^2+\alpha+1)^2}{4\rho^3(x)}\end{aligned} \tag{66}$$

and the following eigenfunctions

$$\psi_0^-(x) = C_0 \frac{\rho(x)+\alpha}{\rho(x)-1} \exp\left[-\frac{1}{2}\rho(x)\left(1+\frac{1}{\alpha}+\frac{1}{\alpha+1}\right)\right] \tag{67}$$

$$\psi_1^-(x) = C_1 \frac{x}{\rho(x)-1} \exp\left[-\frac{1}{2}\rho(x)\left(1-\frac{1}{\alpha}+\frac{1}{\alpha+1}\right)\right] \tag{68}$$

$$\psi_2^-(x) = C_2 \frac{\rho(x)-\alpha-1}{\rho(x)-1} \exp\left[-\frac{1}{2}\rho(x)\left(1-\frac{1}{\alpha}-\frac{1}{\alpha+1}\right)\right] \tag{69}$$

corresponding to the levels

$$E_0^- = 0 \qquad E_1^- = \epsilon_1 + \tfrac{3}{2}b^2 \qquad E_2^- = 2\epsilon_1 + \tfrac{3}{2}b^2 \tag{70}$$

where

$$\rho(x) = \sqrt{1+\alpha(\alpha+1)(1+b^2x^2)} \qquad \alpha = 1 + 2\frac{\epsilon_1}{b^2}.$$

The obtained potential (66) has one minimum at $x = 0$ and tends to a constant for $x \to \pm\infty$. For the case $\alpha > 1$ all the obtained wavefunctions (67)–(69) are square integrable and there exist at least three bound states in the well. For $(\sqrt{5}-1)/2 < \alpha \leqslant 1$ only two lower wavefunctions (67) and (68) are square integrable and we have just two bound states, whereas for $\alpha \leqslant (\sqrt{5}-1)/2$ only the ground-state wavefunction (67) remains square integrable and the potential has only one bound state.

5. Constructing exactly solvable potentials

Although we have developed our scheme to construct new QES potentials it seems to be also of use for constructing SUSY partner potentials for the exactly solvable ones. Let us start from some exactly solvable potential $V_-(x)$

$$V_-(x) = \tfrac{1}{2}(W^2(x) - W'(x)) \tag{71}$$

for which we know three first superpotentials $W(x)$, $W_1(x)$ and $W_2(x)$ satisfying the set of equations (21). One can easily construct the functions $W_+(x) = W(x) + W_1(x)$, $\tilde{W}_+(x) = W_1(x) + W_2(x)$ and $U(x) = W_+(x)\tilde{W}_+(x)$. Substituting function $U(x)$ into the expressions (31), we can obviously reproduce the functions $W_+(x)$ and $\tilde{W}_+(x)$ by choosing the corresponding signs in the function (32) which we shall denote as $\mathcal{R}_0(x)$. Besides the functions $W_+(x)$ and $\tilde{W}_+(x)$, we can obviously obtain another pair of functions $\mathcal{W}_+(x)$ and $\tilde{\mathcal{W}}_+(x)$ given by the same expressions (31) with the only difference being that here we choose function (32) with the opposite sign to that of $\mathcal{R}_0(x)$,

$$\mathcal{R}(x) = -\mathcal{R}_0(x). \tag{72}$$

The new functions $\mathcal{W}_+(x)$ and $\tilde{\mathcal{W}}_+(x)$ satisfy the same equation (26) and they allow us to construct new exactly solvable potentials.

Let us consider these functions in more detail. It is easy to show that both functions $\mathcal{W}_+(x)$ and $\tilde{\mathcal{W}}_+(x)$ are negative at infinity and positive at minus infinity. Explicit calculations show that they provide the same behaviour of the superpotentials $\mathcal{W}(x)$, $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ which can be obtained by substitution of functions $\mathcal{W}_+(x)$ and $\tilde{\mathcal{W}}_+(x)$ into relations (24). We shall omit explicit expressions for the superpotentials $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ noting only that both of them are singular while the superpotential

$$\mathcal{W}(x) = \frac{U'(x)(\mathcal{R}_0(x)-1)}{2(2\epsilon + U(x))} + \frac{8\epsilon\epsilon_1 - U''(x) + 2U(x)(4(\epsilon_1 - \epsilon) - 3U(x))}{2U'(x)\mathcal{R}_0(x)} \tag{73}$$

has no singularities if only the initial potential $V_-(x)$ is non-singular. Using the superpotential $\mathcal{W}(x)$ we can construct in a standard way the pair of Hamiltonians $\mathcal{H}_-$ and $\mathcal{H}_+$. Their properties will be very similar to that of the Hamiltonians H_- and H_+ with the only difference being that now the Hamiltonian $\mathcal{H}_+$ will have zero-energy ground state with the corresponding eigenfunction

$$\varphi_0^+(x) = C\exp\left(\int \mathcal{W}(x)\,\mathrm{d}x\right). \tag{74}$$

All the higher eigenvalues of the Hamiltonians $\mathcal{H}_+$ and $\mathcal{H}_-$ will coincide and the corresponding eigenfunctions will be connected as follows:

$$\varphi_{n+1}^+(x) = \frac{1}{\sqrt{2\mathcal{E}_n^-}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{W}(x)\right)\varphi_n^-(x) \tag{75}$$
$$\varphi_n^-(x) = \frac{1}{\sqrt{2\mathcal{E}_{n+1}^+}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{W}(x)\right)\varphi_{n+1}^+(x).$$

Let us consider a few simple explicit examples. In the case of a harmonic oscillator for which all the superpotentials and distances between the energy levels read

$$W_n(x) = \epsilon x \qquad \epsilon_n = \epsilon \tag{76}$$

we find the corresponding function

$$U(x) = 4\epsilon^2 x^2 \tag{77}$$

and we obtain

$$\mathcal{R}_0(x) = R(x) = 2\epsilon x^2. \tag{78}$$

Choosing now $\mathcal{R}(x) = -\mathcal{R}_0(x)$ and using (73), we obtain

$$\mathcal{W}(x) = -\epsilon x \frac{5 + 2\epsilon x^2}{1 + 2\epsilon x^2} \tag{79}$$

which leads to the following SUSY partner potentials

$$\mathcal{V}_-(x) = \frac{\epsilon^2 x^2}{2} + \frac{5\epsilon}{2} \tag{80}$$

$$\mathcal{V}_+(x) = \frac{\epsilon^2 x^2}{2} + \frac{4\epsilon}{1 + 2\epsilon x^2} - \frac{8\epsilon}{(1 + 2\epsilon x^2)^2} + \frac{3\epsilon}{2}. \tag{81}$$

Thus we have obtained the exactly solvable potential (81) which is a SUSY partner to the harmonic oscillator potential (80). This potential (81) after substituting $\epsilon = b^2/2$ coincides with the potential (63) obtained in section 3. Let us recall that the potential $\mathcal{V}_+(x)$ has a zero-energy level with the corresponding wavefunction (74). Therefore, we can now treat the potentials $\mathcal{V}_-(x)$ and $\mathcal{V}_+(x)$ as $V_+(x)$ and $V_-(x)$, respectively, that is $V_+(x) = \mathcal{V}_-(x)$ and $V_-(x) = \mathcal{V}_+(x)$. The superpotential corresponding to the pair $V_-(x)$ and $V_+(x)$ is $W(x) = -\mathcal{W}(x)$. Because the upper SUSY partner $V_+(x)$ is a harmonic oscillator we can easily build up all the hierarchy of superpotentials satisfying equations (21):

$$W_0(x) = \epsilon x \frac{5 + 2\epsilon x^2}{1 + 2\epsilon x^2} \qquad W_n(x) = \epsilon x \qquad n = 1, 2, \ldots. \tag{82}$$

The corresponding distances between the energy levels read

$$\epsilon_0 = 3\epsilon \qquad \epsilon_n = \epsilon \qquad n = 1, 2, \ldots. \tag{83}$$

The obtained potential (81) can be used to construct another exactly solvable potential. Starting from the first three superpotentials (82) we obtain in the same way as before the next pair of potentials

$$\mathcal{V}_-(x) = \frac{\epsilon^2 x^2}{2} + \frac{9\epsilon}{2} \tag{84}$$

$$\mathcal{V}_+(x) = \frac{\epsilon^2 x^2}{2} + \frac{8\epsilon(2\epsilon x^2 - 3)}{3 + 12\epsilon x^2 + 4\epsilon^2 x^4} + \frac{384\epsilon^2 x^2}{(3 + 12\epsilon x^2 + 4\epsilon^2 x^4)^2} + \frac{7\epsilon}{2}. \tag{85}$$

Repeating this procedure many times we obtain the following pairs of SUSY partner potentials

$$V_+(n, x) = \mathcal{V}_-(n, x) = \frac{\epsilon^2 x^2}{2} + \left(2n - \frac{1}{2}\right)\epsilon. \tag{86}$$

$$\begin{aligned} V_-(n, x) = \mathcal{V}_+(n, x) = {} & \frac{\epsilon^2 x^2}{2} + 8\epsilon n(2n - 1)\frac{H_{2n-2}(\mathrm{i}\sqrt{\epsilon}x)}{H_{2n}(\mathrm{i}\sqrt{\epsilon}x)} \\ & -16\epsilon n^2 \left(\frac{H_{2n-1}(\mathrm{i}\sqrt{\epsilon}x)}{H_{2n}(\mathrm{i}\sqrt{\epsilon}x)}\right)^2 + \frac{(4n - 1)\epsilon}{2} \end{aligned} \tag{87}$$

where $H_n(x)$ is the Hermite polynomial. Note that in the case $n = 1$ the potential (87) which corresponds to (81) and in the case $n = 2$ it corresponds to (85). The potentials $V_-(n, x)$ (87) are just the special cases of the SUSY partner potential of a harmonic oscillator obtained by

Sukumar [21] and they were previously obtained by Bagrov and Samsonov [22, 23] via the Darboux method and latter by Junker and Roy [33] within the SUSY approach.

The application of the same procedure for the Morse and Rosen–Morse potentials as well as for the radial harmonic oscillator and the hydrogen atom provides chains of exactly solvable potentials. All these potentials are just special cases of the potentials obtained in [33] by Junker and Roy. Nevertheless, they seem to be interesting because they are all expressed in terms of the elementary functions only. For example, starting from the Morse potential for which corresponding superpotentials and distances between the energy levels read

$$W_n(x) = \epsilon + 1/2 - n - \mathrm{e}^{-x} \qquad \epsilon_n = \epsilon - n \tag{88}$$

we construct the function

$$U(x) = 4(\epsilon - \mathrm{e}^{-x})(\epsilon - 1 - \mathrm{e}^{-x}). \tag{89}$$

An explicit calculation shows us that we should take the function $\mathcal{R}_0(x)$ in the form

$$\mathcal{R}_0(x) = \frac{4\,\mathrm{e}^{-2x} + 6(1-2\epsilon)\,\mathrm{e}^{-x} + 3(1+4\epsilon(\epsilon-1)) - 2\epsilon(1-3\epsilon+2\epsilon^2)\,\mathrm{e}^{x}}{2\,\mathrm{e}^{-x} + (1-2\epsilon)} \tag{90}$$

to reproduce the Morse potential. The function $\mathcal{R}_0(x)$ does not coincide with $R(x)$ in this case, the former is negative within the interval limited by zeros of expression (90). Following the same procedure as in the case of the harmonic oscillator, we obtain such a sequence of potentials

$$V_-(0,x) = \frac{(1+2\epsilon)^2}{8} + \frac{\mathrm{e}^{-2x} - 2(\epsilon+1)\,\mathrm{e}^{-x}}{2} \tag{91}$$

$$\begin{aligned} V_-(1,x) = {} & \frac{(1+2\epsilon)^2}{8} + \frac{\mathrm{e}^{-2x} - 2(\epsilon-1)\,\mathrm{e}^{-x}}{2} + \frac{2(\epsilon(2\epsilon-1)\,\mathrm{e}^{x} - 2)}{\epsilon(2 - 2(2\epsilon-1)\,\mathrm{e}^{x} + \epsilon(2\epsilon-1)\,\mathrm{e}^{2x})} \\ & - \frac{8((2\epsilon-1)\,\mathrm{e}^{x} - 1)}{\epsilon(2 - 2(2\epsilon-1)\,\mathrm{e}^{x} + \epsilon(2\epsilon-1)\,\mathrm{e}^{2x})^2} \end{aligned} \tag{92}$$

$$\begin{aligned} V_-(2,x) = {} & \frac{(1+2\epsilon)^2}{8} + \frac{\mathrm{e}^{-2x} - 2(\epsilon-3)\,\mathrm{e}^{-x}}{2} \\ & + \frac{(2\epsilon-3)(8\,\mathrm{e}^{x} - (\epsilon-1)(48\,\mathrm{e}^{2x} - (2\epsilon-1)(36\,\mathrm{e}^{3x} - 16\epsilon\,\mathrm{e}^{4x})))}{4 - (2\epsilon-3)(8\,\mathrm{e}^{x} - (\epsilon-1)(12\,\mathrm{e}^{2x} - (2\epsilon-1)(4\,\mathrm{e}^{3x} - \epsilon\,\mathrm{e}^{4x})))} \\ & + \frac{((2\epsilon-3)(8\,\mathrm{e}^{x} - (\epsilon-1)(24\,\mathrm{e}^{2x} - (2\epsilon-1)(12\,\mathrm{e}^{3x} - \epsilon 4\,\mathrm{e}^{4x}))))^2}{(4 - (2\epsilon-3)(8\,\mathrm{e}^{x} - (\epsilon-1)(12\,\mathrm{e}^{2x} - (2\epsilon-1)(4\,\mathrm{e}^{3x} - \epsilon\,\mathrm{e}^{4x}))))^2}. \end{aligned} \tag{93}$$

The corresponding SUSY partners are the following Morse potentials

$$V_+(n,x) = \frac{(1+2\epsilon)^2}{8} + \frac{\mathrm{e}^{-2x} - 2(\epsilon-2n)\,\mathrm{e}^{-x}}{2}. \tag{94}$$

We can proceed with this procedure as long as necessary.

The most interesting fact is that at each step of the suggested procedure for all the above-mentioned potentials

$$\mathcal{H}_- = H_+^{(2)} + \epsilon + \epsilon_1. \tag{95}$$

One can easily check that due to the connection (11) and (20)

$$E_n^{(2)+} = E_{n+1}^{(2)-} = E_{n+1}^{(1)+} - \epsilon_1 = E_{n+2}^{+} - \epsilon_1 - \epsilon = E_{n+3}^{-} - \epsilon_1 - \epsilon \tag{96}$$

and therefore the energy levels of the Hamiltonian $\mathcal{H}_-$ coincide with that of the Hamiltonian H_- saving the three lowest levels of the latter which are not present in the spectrum of the former

$$\mathcal{E}_n^- = E_{n+3}^- \qquad n = 0, 1, \ldots. \tag{97}$$

This gives us immediately the energy spectrum of the new Hamiltonian $\mathcal{H}_+$

$$\mathcal{E}_0^- = 0 \qquad \mathcal{E}_n^- = E_{n+2}^- \qquad n = 1, 2, \ldots. \tag{98}$$

Moreover, the connection (95) allows us to easily obtain all the eigenfunctions of the excited states for the new exactly solvable potential $\mathcal{H}_+$. Using relation (75) we obtain

$$\varphi_n^+(x) = C_n^+ \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{W}(x) \right) \psi_{n-1}^{(2)+}(x) \qquad n = 1, 2, \ldots. \tag{99}$$

Thus for all the mentioned exactly solvable potentials we can construct a sequence of Hamiltonians $H_n = -\frac{1}{2}\mathrm{d}^2/\mathrm{d}x^2 + V(n, x)$ with the same energy levels as H_0 except the $2n$ lowest excited states of the latter. They all possess the zero-energy ground state.

6. Conclusions

We have obtained a general solution of the set of equations (21) for $N = 2$ given by the expressions (31) and (24) ($n = 0, 1$). Thus we can write down explicit expressions for the superpotentials $W_0(x)$, $W_1(x)$ and $W_2(x)$ and then for the potential $V_-^{(0)}(x) = (W_0^2(x) - W_0'(x))/2$. It will be just a general expression for a QES potential with three eigenstates which are explicitly known. General expressions for the corresponding eigenfunctions are also presented, (9) and (35). The QES potential is expressed in terms of the distances ϵ and ϵ_1 between neighbouring energy levels and an arbitrary function $U(x)$. To ensure non-singularity of the potential we need to put a number of limitations on the function $U(x)$. Using this expression we have obtained some new QES potentials. In special cases our potentials reproduce those studied earlier.

There obviously arises a question as to whether the suggested scheme could be generalized to construct QES potentials when the number of explicitly known eigenstates is larger than three. In such a case we have the set of equations (25) consisting of $N - 2$ equations. Let us recall that N is the number of known excited states of the QES potential which we would like to construct. In order to reduce this set of equations we can proceed with the scheme described in section 3 (equations (21)–(31)).The solution of each equation of the set (25) can be written down in the form (31), where $W_+(x)$ is replaced by $W_+^{(n)}(x)$, $\tilde{W}_+(x)$ is replaced by $W_+^{(n+1)}(x)$ and we have $U_n(x) = W_+^{(n)}(x)W_+^{(n+1)}(x)$ instead of $U(x)$. Two neighbouring equations of the set (25) yield two different expressions for the same function $W_+^{(n)}$ in terms of the functions $U_n(x)$ and $U_{n+1}(x)$ correspondingly. Thus, we obtain the following set of equations for the functions $U_n(x)$

$$\frac{U_n'(x)\,(1 + \mathcal{R}_n(x))}{2(U_n(x) + 2\epsilon_n)} = \frac{2U_{n+1}(x)(U_{n+1}(x) + 2\epsilon_{n+1})}{U_{n+1}'(x)\,(1 + \mathcal{R}_{n+1}(x))} \qquad n = 0, \ldots, N-3 \tag{100}$$

where

$$\mathcal{R}_n(x) = \pm\sqrt{1 + 4\frac{U_n(x)(U_n(x) + 2\epsilon_n)(U_n(x) - 2\epsilon_{n+1})}{U_n'(x)^2}}. \tag{101}$$

The number of equations in this set is one less than the number of equations in the set (25) for the functions $W_+^{(n)}(x)$ and correspondingly it is two less than the number of initial equations (21) for the superpotentials. In the case $N = 3$ we have just one equation which we need to solve to construct QES potentials with four known eigenstates. Thus, one can see that to obtain the general expression for QES potentials with more than three explicitly known eigenstates is essentially more complicated.

Another point is that the suggested scheme allows one to construct in a simple way the sequences of SUSY partner potentials of exactly solvable ones. At each step we obtain a new exactly solvable potential with an identical spectrum, apart from missing the two lowest excited states. Note that this approach, in contrast to the method used in [20–23, 32, 33], does not require knowledge of the general solution of the corresponding Schrödinger equation for the initial potential. It would be interesting to apply the same approach to the known QES potentials with explicitly known N eigenstates to construct new QES potentials with the two lowest excited states picked out. However, the latter case is more complicated than the case of shape-invariant potentials and will be the subject of a separate paper.

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